Second Semester [B.Tech]
Applied Mathematics - II
Model Test Paper

Q1. ⓐ Find poles, its order and residue of the function $f(z) = \dfrac{1-e^{2z}}{z^4}$.

Hint: $f(z) = \dfrac{1}{z^4}\left[1-e^{2z}\right] = -\left[\dfrac{2}{z^3} + \dfrac{2}{z^2} + \dfrac{4}{3z} + \dfrac{2}{3} + \dfrac{4}{15}z + \cdots\right]$

then use Laurent's series

ⓑ Expand $\dfrac{1}{z(z^2-3z+2)}$ for the region $1 \le |z| < 2$

Hint: $f(z) = \dfrac{1}{2z} - \dfrac{1}{z-1} + \dfrac{1}{2(z-2)}$ by using partial fraction.

$= -\dfrac{1}{2z} - \dfrac{1}{z^2} - \dfrac{1}{z^3} - \dfrac{1}{z^4} \cdots - \dfrac{1}{4}\left[1 + \dfrac{z}{2} + \dfrac{z^2}{4} + \dfrac{z^3}{8} + \cdots\right]$

ⓒ Evaluate $\int_C e^{1/z^2}\,dz$ where $C: |z| = 2$

Hint: $f(z)$ has isolated singularity at $z=0$ use Laurent's series expansion to get residue

ⓓ Find the directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ at the point $P(1,2,3)$ in the direction of the line PQ where $Q(5,0,4)$

Hint find $\nabla\phi|_P = 2\hat{i} - 4\hat{j} + 12\hat{k}$

$\vec{PQ} = 4\hat{i} - 2\hat{j} + \hat{k}$

find directional derivative of ϕ along $\vec{PQ}$

$= \nabla\phi \cdot \hat{PQ} = \dfrac{28}{\sqrt{21}}$

ⓔ Prove that $\text{Div}(\text{grad}\,\phi) = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}$

Hint: $\nabla\cdot(\nabla\phi) = \nabla^2\phi = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}$

$$\nabla^2\phi = (\nabla\cdot\nabla)\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

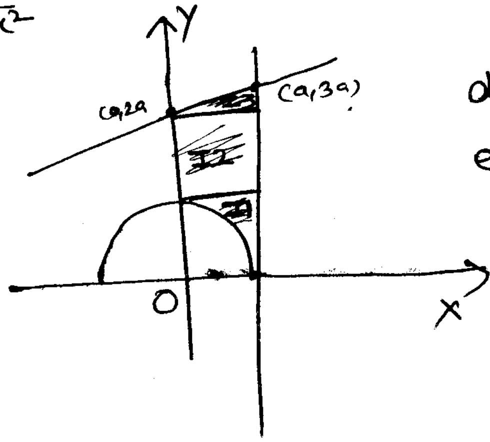
$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

(f) Change the order of integration ...

$$\int_0^a \int_{\sqrt{a^2-x^2}}^{x+2a} f(x,y)\,dx\cdot dy$$

Hint:



draw curves of four equations

$y=\sqrt{a^2-x^2}$ $\quad x=0$

$y = x+2a$ $\quad x=a$

Q2 (a) Form the partial Differential equation by eliminating the arbitrary function f from $f(xy+z^2,\ x+y+z) = 0$

HINT: Write $f(u,v) = 0$ — (1)

where $u = xy+z^2$, $V = x+y+z$

Differentiating (1) partially w.r.t x & y

& then eliminating $\frac{\partial f}{\partial u}$ & $\frac{\partial f}{\partial v}$.

to get Differential eqn

$$p(x-2z) + q(2z-y) = y-x$$

(b) If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ then show that

$$x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$$

HINT: $\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = f$.

so f is homogeneous fn of degree 1/2

& hence so is $\cos u$

then use Euler's Thm

$$x\cdot\frac{\partial}{\partial x}\cos u + y\cdot\frac{\partial}{\partial y}\cos u + z\cdot\frac{\partial}{\partial z}\cos u = \frac{1}{2}\cos u.$$

Q3 (a) Prove that the function

$u = x+y+z$, $v = x^3+y^3+z^3-3xyz$

$w = x^2+y^2+z^2-xy-yz-zx$ are functionally dependent. find relation between them.

HINT: The functions u, v, w are functionally dependent if $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

(b) The temperature T at any point (x, y, z) of space is given by. $T = 400xyz^2$ find the highest temperature at the surface of the sphere $x^2+y^2+z^2 = 1$

HINT: $T = 400xyz^2$

where $x^2+y^2+z^2 = 1$.

$T' = xyz^2 = xy(1-x^2-y^2)$

$\frac{\partial T'}{\partial x} = 0$ & $\frac{\partial T'}{\partial y} = 0$

$\Rightarrow$ $x = 0$, $y = 0$

$x = \frac{1}{2}$, $y = \frac{1}{2}$

$x = \frac{1}{2}$, $y = -\frac{1}{2}$

$x = -\frac{1}{2}$, $y = \frac{1}{2}$ are critical points

Calculate $rt - s^2$ where $r = \frac{\partial^2 T'}{\partial x^2}$

$t = \frac{\partial^2 T'}{\partial y^2}$, $s = \frac{\partial^2 T'}{\partial x \cdot \partial y}$.

Q4 (a) State Residue Theorem & use it to evaluate $\int_C \frac{dz}{z^8(z+4)}$ where

$C: |z| = 2$

HINT :- $f(z)$ has singularities at $z = 0$ & $z = -4$

Only $z = 0$ lies inside C

calculate residue at $z = 0$ which is pole of order 8 & then apply residue Thm. to get $\frac{-2\pi i}{4^8}$ as answer.

(b) Find the transformation ~~of~~ which maps the points $z = 1, -i, -1$ to the points $\omega = i, 0, -i$ respectively. Show that this transformation maps the region outside the circle $|z| = 1$ into the half space $R(\omega) \geq 0$

HINT :- The bilinear transformation is

$$\frac{(\omega-\omega_1)(\omega_2-\omega_3)}{(\omega-\omega_3)(\omega_2-\omega_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \quad \omega = \frac{z+i}{z-i}$$

Q5. (a) Determine the analytic function $f(z) = u + iv$ if $v = \log(x^2+y^2) + x - 2y$

Sol^n $\because$ $v = \log(x^2+y^2) + x - 2y$

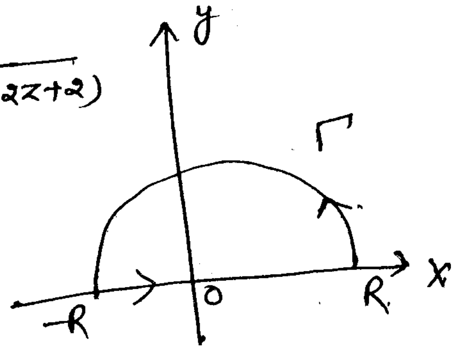
$$\frac{\partial v}{\partial x} = \frac{2x}{x^2+y^2} + 1 \qquad \frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2} - 2$$

use C.R. equations

$\therefore$ $f'(z) = u_x + iv_x \Rightarrow$ then apply Milne Thomson's Method $x \to z$, $y \to 0$

Q5 (b) Show that $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} \frac{dx}{(x^2+2x+2)} = \frac{7\pi}{50}$

HINT: Poles of $f(z) = \frac{z^2}{(z^2+1)^2(z^2+2z+2)}$



$z = \pm i$, $z = -1 \pm i$

only $z = i$, $z = -1+i$

of order 2 & order 1

respectively lies inside C

where C consist of semi circle Γ.

& real axis $-R$ to R

Calculate Residue at $z = i$ & $z = -1+i$

& find integral.

Q6 (a) Verify Gauss divergence thm for

$\vec{F} = y\hat{i} + x\hat{j} + z^3\hat{k}$ taken over the

cylindrical region $x^2+y^2 = 9$; $z=0$ to $z=6$

HINT: $\iint_S \vec{F} \cdot \vec{ds} = \iint_I (y\hat{i}+x\hat{j}+z^3\hat{k}) \cdot \frac{x\hat{i}+y\hat{j}}{3} ds +$

$\iint_{II} (y\hat{i}+x\hat{j}+0^3\hat{k}) (-\hat{k}\, ds) + \iint_{III} (y\hat{i}+x\hat{j}+6^3\hat{k}) \hat{k}\, ds$

I: curved surface

II: circular plate at $z=0$

III: circular plate at $z=6$

(b) use Green's thm in the plane

evaluate $\int [\cos y\,\hat{i} + x(1-\sin y)\hat{j}] \cdot \vec{dr}$

for a closed curve given by

$x^2+y^2 = 1$, $z = 0$

HINT: By Green's Thm $\oint u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$

where C is closed curve enclosing area R.

$$\int \vec{F} \cdot \vec{dr} = \iint_R (\nabla \times \vec{F}) \cdot \vec{ds}$$

$$\int \vec{F} \cdot \vec{dr} = \iint_R K \cdot \vec{ds} = \iint_R K \cdot K \, dx\,dy$$

$$= \iint dx\,dy = \pi$$

Q7 (a) Evaluate $\oint \vec{F} \cdot \vec{dr}$ by stoke's thm

Where $F = y^2\hat{i} + x^2\hat{j} - (x+z)\,y^2\hat{k}$ &

C is boundary of the triangle with vertices $(0,0,0)$, $(1,0,0)$ & $(1,1,0)$.

HINT: By stoke's thm $\int_C \vec{F} \cdot \vec{dr} = \iint_S (\nabla \times \vec{F}) \cdot \vec{ds}$

$$\iint (\nabla \times \vec{F})\, dx\,dy = 1/3$$

(b) Prove that $\vec{F} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational. find scalar ϕ such that $\vec{F} = \nabla \phi$

HINT:

If $\nabla \times \vec{F} = 0$ then $\vec{F}$ is irrotational

Let $\nabla \phi = \vec{F}$

$$\Rightarrow \frac{\partial \phi}{\partial x} = 6xy + z^3 \qquad \frac{\partial \phi}{\partial y} = 3x^2 - z, \qquad \frac{\partial \phi}{\partial z} = 3xz^2 - y$$

find $\phi = 3x^2y + xz^3 - yz$

Q8 (a) Use convolution Thm to evaluate the Laplace transform of $\frac{s^2}{(s^2+\omega^2)^2}$

Solⁿ $L^{-1}\left[\frac{s^2}{(s^2+\omega^2)^2}\right] = L^{-1}\left[\frac{s}{s^2+\omega^2} \times \frac{s}{s^2+\omega^2}\right]$

$$= \int_0^t \cos\omega u \cos\omega(t-u)\, du$$

$$= \frac{1}{2}\int_0^t [\cos\omega t + \cos\omega(t-2u)]\, du$$

$$= \frac{1}{2}\left[t\cos\omega t + \frac{1}{\omega}\sin\omega t\right]$$

$$L^{-1}\left[\frac{s^2}{(s^2+\omega^2)^2}\right] = \frac{t}{2}\cos\omega t + \frac{1}{\omega}\sin\omega t$$

(b) Prove that $L\left[\int_0^t e^t \frac{\sin t}{t}\, dt\right] = \frac{1}{s}\cot^{-1}(s-1)$

HINT: $L(e^t \sin t) = \frac{1}{(s-1)^2+1}$

$$L\left[\frac{e^t \sin t}{t}\right] = \int_s^\infty \frac{ds}{(s-1)^2+1} = \cot^{-1}(s-1)$$

$$\therefore L\left[\int_0^t \frac{e^t \sin t}{t}\, dt\right] = \frac{1}{s}\cot^{-1}(s-1)$$

Q9 (a) Using Laplace transform, to solve the eqⁿ $y'' + y = 6\cos 2t$

$y(0) = 3 \qquad y'(0) = 1$

HINT: Taking Laplace transform both side

$$s^2\bar{y} - sy(0) - y'(0) + \bar{y} = \frac{6s}{s^2+4}$$

$$\Rightarrow \bar{y} = \frac{5s+1}{s^2+1} - \frac{2s}{s^2+4}$$

$$\Rightarrow y = L^{-1}\left[\frac{5s+1}{s^2+1} - \frac{2s}{s^2+4}\right] = 5\cos t + \sin t - 2\cos 2t$$

Q9 (b) using unit step function and dirac delta function find

$$L\left[t\,U(t-4) - t^3\,\delta(t-2)\right]$$

HINT:- $L[t\,U(t-4)] = L[(t-4)U(t-4)] + 4L[U(t-4)]$

$$= e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right)$$

$$L[t^3\,\delta(t-2)] = f(2)\,e^{-2s} = 8\,e^{-2s}$$

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