Model Test Paper

II Sem [B. Tech] 2014 Max Marks - 75

Applied Mathematics - II Time - 3 hour

Note:- Attempt one Question from each unit. Q.No. 1 is Compoulsory.

Q.1 (a) If $x^y + y^x = c$, find $\frac{dy}{dx}$ (3)

Soln Hint: $f(x,y) = x^y + y^x = c$

then $\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y} = \frac{-yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$

(3)

(b) Prove that

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$$

(c) Compute an approximate value of $(1.04)^{3.01}$. (3)

Soln Hint: Let $f(x,y) = x^y$, Let $x = 1$, $dx = 0.04$, $y = 3$, $dy = 0.01$

and $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \Rightarrow df = 0.12$

$\therefore (1.04)^3 = f(1,3) + df = 1.12$

(d) Find the stationary points of the function (3)

$2(x-y)^2 - x^4 - y^4$

Hint: Let $f = 2(x-y)^2 - x^4 - y^4$

Solving $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ for x and y, we get

$(0,0)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$

(e) If θ is the acute angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1, -2, 1)$, show that $\cos\theta = 3/7\sqrt{6}$

Hint: Let $\phi_1 = xy^2z - 3x - z^2 = 0$ [3]

$\phi_2 = 3x^2 - y^2 + 2z = 1$

$$\nabla\phi_1 \cdot \nabla\phi_2 = |\nabla\phi_1| \cdot |\nabla\phi_2| \cos\theta$$

$$\Rightarrow \cos\theta = 3/7\sqrt{6}$$

(f) Obtain Laurent's series for the function $f(x) = \dfrac{1}{z^2 \sinh z}$ at the isolated singularity [4]

Hint:
$$\frac{1}{z^2 \sinh z} = \frac{1}{z^2\left[z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right]}$$
$$= \frac{1}{z^3} - \frac{1}{6z} + \frac{7}{360}z^4 - \cdots .$$

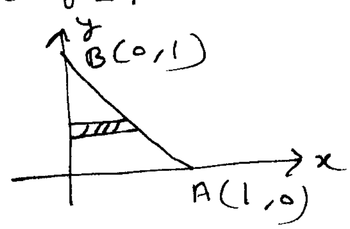
(g) Find the Laplace transform of (3)
$(t-1)^2 u(t-1)$

Ans: $e^{-s} \cdot \dfrac{2!}{s^3}$

(h) Evaluate $\iint_R (x^2+y^2)\,dx\,dy$ over the region in the positive quadrant for which $x+y \leq 1$. (3)

Hint: $\iint_R (x^2+y^2)\,dx\,dy = \int_0^1 \int_0^{1-y} (x^2+y^2)\,dx\,dy$

$$= \frac{1}{6}$$



Unit - I

Q.2(a) Given $\theta = t^n \cdot e^{-r^2/4t}$, find the value of n, for which $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = \dfrac{\partial \theta}{\partial t}$. (6.5)

Ans: $n = -3/2$

Q.2(b) If $u = \sin^{-1}\left[\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right]^{1/2}$, (6.5)

Show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}\left(13+\tan^2 u\right)$$

Hint: Use Euler's thm

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

where $z = \sin u = \left[\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right]^{1/2}$ and $n = -\dfrac{1}{12}$

Q.3(a) If $u = \dfrac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$, (6)

determine whether there is a functional relationship between u and v, if so, find it.

Hint: Prove $\left[\dfrac{\partial(u,v)}{\partial(x,y)} = 0\right]$, $u = \tan v$ (relation)

3(b) Show that the maximum and minimum values of r^2, where $r^2 = a^2x^2+b^2y^2+c^2z^2$, and $x^2+y^2+z^2=1$, $lx+my+nz=0$, are given by the equation (6.5)

$$\frac{l^2}{a^2-r^2} + \frac{m^2}{b^2-r^2} + \frac{n^2}{c^2-r^2} = 0$$

Hint: Use Lagrange's method of undetermined multipliers.

Unit - II

Q.4 (a) If $f(z)$ is a regular function of z, Prove that (6)

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

Hint: $f(z) = u+iv$, $|f(z)| = \sqrt{u^2+v^2} = F$ (say)

find $\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}$

Q.4 (b) Under the transformation $w=\frac{1}{z}$, the image of the hyperbola $x^2-y^2=1$ is the lemniscate $\rho^2 = \cos 2\phi$ (6.5)

Hint: $u+iv = \frac{1}{x+iy}$

$\Rightarrow x+iy = \frac{1}{u+iv}$

$\Rightarrow x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$

or

$w = Re^{i\phi} = \frac{1}{re^{i\theta}} \Rightarrow R = \frac{1}{r}$ and $\phi = -\theta$

Q. 5 (a) Evaluate (6)

$$\int \frac{(12z-7)}{(z-1)^2(2z+3)}\,dz, \text{ where } c: |z| = 2$$

Hint: use $\int f(z)\,dz = 2\pi i R^+$ (sum of residues)

Here $f(z)$ has $z=1$ (double pole) and $z=3/2$ (single pole)

both the poles lie inside the given

circle. Use the formulae to find the residues; i.e.

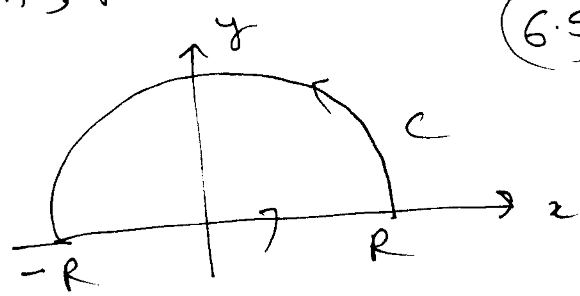
$$\text{Res}\{f(z)\}_{z=1} = \frac{1}{1!} \lim_{z\to 1}\left[\frac{d}{dz} f(z)\cdot(z-1)^2\right]$$

and $\text{Res}\{f(z)\}_{z=3/2} = \lim_{z\to 3/2} (z-3/2) f(z)$.

Ans :- $-4\pi i$

Q. 5(b) Using residue theorem, prove that (6.5)

$$\int_0^\infty \frac{dx}{x^4+a^4} = \frac{\pi\sqrt{2}}{4a^3}$$



Hint: Let $\phi(z) = \dfrac{1}{z^4+a^4}$

poles of $\phi(z)$ are $= z^4 = -a^4 = e^{(2n+1)\pi i} a^4$

$n = 0, 1, 2, 3$

or $z = ae^{\frac{\pi i}{4}}, \; ae^{\frac{3\pi i}{4}}, \; ae^{5\pi i/4}, \; ae^{\frac{7\pi i}{4}}$

out of which $z = ae^{\pi i/4}$, $ae^{3\pi i/4}$ lie with in C.

$$\text{Res}\{\phi(z)\}_{z=ae^{\pi i/4}} = -e^{\pi i/4}/4a^3$$

and $\text{Res}\{f(z)\}_{z=ae^{3\pi i/4}} = -e^{3\pi i/4}/4a^3$

Unit - III

6(a) Apply convolution theorem to evaluate (6)

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$$

Hint: $L^{-1}\{\bar{f}(s)\,\bar{g}(s)\} = \int_0^t f(u)\, g(t-u)\, du$, where

$f(t) = \cos at$ and $g(t) = \cos bt$.

Q.6(b) Solve the following

(i) $L^{-1}\left\{\log\left(\frac{s+1}{s-1}\right)\right\}$ Ans:- $\frac{2\sinh t}{t}$ (6.5)

(ii) $L^{-1}\left\{\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}\right\}$ Ans:- $\left\{u\left(t-\frac{1}{2}\right)-u(t-1)\right\} \times \sin \pi t$

Q.7(a) Solve by Laplace transformation (6.5)

$$\frac{d^2y}{dx^2}+2\frac{dy}{dt}+2y=\frac{17}{2}\sin 5t,$$

$$y(0)=2,\quad y'(0)=-4$$

Hint: $L\{y^n(x)\} = s^n\tilde{y} - s^{n-1}y(0) - s^{n-2}y'(0) + \cdots + y^{n-1}(0)$.

Q.7(b) Evaluate the integral (6)

(i) $\int_0^\infty t e^{-2t}\sin t\,dt$ Ans:- $\frac{4}{25}$

Hint $\int_0^\infty t e^{-2t}\sin t\,dt = L\{t\sin t\}_{s\to 2}$

(ii) $L\left\{\int_0^t \frac{e^t\sin t}{t}\,dt\right\}$

Hint:- $L\left\{\frac{\sin t}{t}\right\} = \cot^{-1}s \Rightarrow L\left\{\frac{e^t\sin t}{t}\right\} = \cot^{-1}(s-1)$

$\therefore L\left\{\int_0^t \frac{e^t\sin t}{t}\,dt\right\} = \frac{\cot^{-1}(s-1)}{s}$

Q. 8(a) i) Prove that $\operatorname{div}(\operatorname{grad} r^n) = n(n+1)r^{n-2}$ (3)

where $r^2 = x^2+y^2+z^2$

(ii) Find the directional derivative of $\nabla\cdot(\nabla\phi)$ at the point $(1,-2,1)$ in the direction of the normal to the surface

$xy^2z = 3x+z^2$, where $\phi = 2x^3y^2z^4$. (3)

Hint: Let $f_1 = \nabla\cdot(\nabla\phi)$

then calculat ∇f_1 at $(1,-2,1)$

Let $f_2 = xy^2z - 3x - z^2$

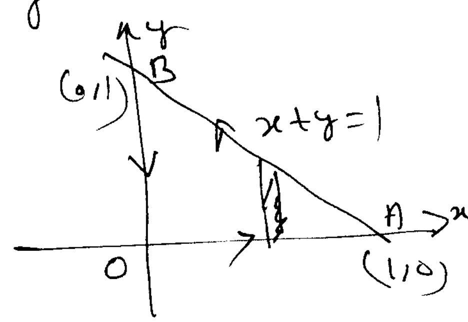
then find $\dfrac{\nabla f_2}{|\nabla f_2|}$.

then required direction derivative will be $(\nabla f_1)_{(1,-2,1)} \cdot \dfrac{\nabla f_2}{|\nabla f_2|}$

$= 1724/\sqrt{21}$

8(b) verify Green's thm for

$\int_C [(3x-8y^2)dx + (4y-6xy)dy]$, where C is the boundary of the region bounded by $x=0$, $y=0$ and $x+y=1$



Hint: $\int_C \phi\,dx + \psi\,dy = \int_{OA} \cdots + \int_{AB} \cdots + \int_{BO} \cdots$

$$\iint_E \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dx\,dy = \int_0^1 \int_{y=0}^{1-x} \cdots$$

where $\phi = 3x - 8y^2$

$\psi = 4y - 6xy$

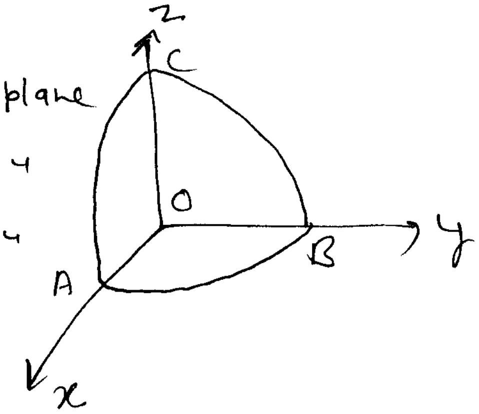
Q. 9(a) Evaluate $\int_S (yz\,i + zx\,j + xy\,k)\cdot ds$, where S is the surface of the sphere (6)

$x^2+y^2+z^2=a^2$ in the first octant

Hint:-

By divergence thm

$\int_V \operatorname{div} F\,dV = \int_{S_1} F\cdot ds + \int_{S_2} F\cdot ds + \int_{S_3} F\cdot ds + \int_S F\cdot ds$



where

S_1 = Circular quadrant OBC in the yz plane

S_2 = " " OCA " " xz "

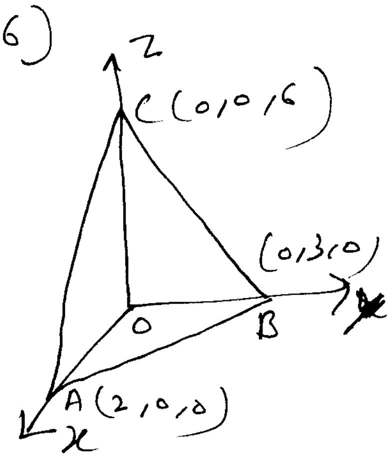
S_3 = " " OAB " " xy "

S = Surface ABC of the sphere in the first octant

Ans:- $\int_S F\cdot ds = 3a^4/8$

Q. 9(b) Using Stoke's thm evaluate $\int_C [(x+y)dx + (2x-z)dy + (y+z)dz]$

where C is the boundary of the triangle with vertices (2,0,0), (0,3,0) and (0,0,6)



Hint: $\int_C F\cdot dR = \int_S \operatorname{curl} F\cdot N\,ds$

$\operatorname{curl} F = 2i + k$,

$N = \dfrac{\nabla F}{|\nabla F|} = \dfrac{3i+2j+k}{\sqrt{14}}$

$\therefore \int_S \operatorname{curl} F\cdot N\,ds = \dfrac{7}{\sqrt{14}}$ Area of $(\triangle ABC) = \dfrac{7}{\sqrt{14}} \times 3\sqrt{14} = 21$ Ans

Ruchika Jain
(Dept. of Mathematics)